G2-continuous composite cubic Bézier curves

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1 Introduction

Consider a given set of points in the plane. How can we draw a smooth curve through them? Stuart Kent [1] outlines a method to construct a C^2-continuous composite cubic Bézier curve through the points. In this paper we extend the method to produce G^2-continuous curves, and we treat also closed curves.

2 Preliminaries

2.1 Composite Bézier curves

The *composite Bézier curves* we shall be working with are continuous curves that are piecewise arcs of cubic Bézier curves. Each Bézier curve is determined by four *control points* p_0, p_1, p_2, p_3, and the arc is defined as

$$B(t) = \sum_{k=0}^{3} b_k^3(t) p_k \qquad (0 \le t \le 1) \tag{1}$$

where $b_k^3(t)$ $(k = 0, \ldots, 3)$ are the *Bernstein polynomials* of degree 3. We call such arcs *Bézier arcs*. The points p_0 and p_3 are *anchors* and the difference vectors $-p_0 + p_1$ and $-p_3 + p_2$ are *handles* (though below we use $-p_2 + p_3$ as a handle).

A composite Bézier curve is a succession of arcs (1) where adjacent Bézier arcs share a common anchor and the directions of the arcs are compatible. We denote the anchors $A_0, \ldots, A_n$. We allow $A_0 = A_n$ in which case the composite Bézier curve is *closed*.

2.2 Derivatives

We list the expression of $B(t)$ in (1) and its derivatives:

$$\begin{cases} B(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t) t^2 p_2 + t^3 p_3, \\ B'(t) = 3\Big[-(1-t)^2 p_0 + (1-t)(1-3t) p_1 + t(2-3t) p_2 + t^2 p_3\Big], \\ B''(t) = 6\Big[(1-t) p_0 + (3t-2) p_1 + (1-3t) p_2 + t p_3\Big]. \end{cases} \tag{2}$$

Denoting $p_{ij} = -p_i + p_j$ and doing some straight-forward calculations, we write B' and B'' at the start point of the Bézier arc,

$$\begin{cases} B'(0) = 3p_{01}, \\ B''(0) = 6(-2p_{01} - p_{23} + p_{03}), \end{cases} \tag{3}$$

and similarly at the end point of the arc,

$$\begin{cases} B'(1) = 3p_{23}, \\ B''(1) = 6(p_{01} + 2p_{23} - p_{03}). \end{cases} \tag{4}$$

2.3 Notations

Assume that we are given in the plane points $A_0, \ldots, A_n$, $n \geq 2$. We assume that $A_i \neq A_{i+1}$ $(i = 1, \ldots, n-1)$. In particular, we allow $A_0 = A_n$ which case means that we are constructing a closed curve. We want to draw a composite Bézier curve having the points $A_0, \ldots, A_n$ as anchors. We choose notations:

- The anchors $A_0, \ldots, A_n$.
- $B_i(t)$ $(0 \leq t \leq 1)$ is a Bézier arc from A_i to A_{i+1} $(i = 0, \ldots, n-1)$.
- The control points of B_i are

$$p_{i0} = A_i,\ p_{i1},\ p_{i2},\ p_{i3} = A_{i+1} \qquad (i = 0, \ldots, n-1).$$

- The handles are

$$\begin{cases} x_i = -p_{i0} + p_{i1} = p_{i,01}, \\ y_i = -p_{i2} + p_{i3} = p_{i,23}, \end{cases} \qquad (i = 0, \ldots, n-1).$$

The vectors x_i and y_i $(i = 0, \ldots, n-1)$ are our unknowns: Since the anchors A_i are given, once we solve the x_i's and y_i's we get the control points of B_i as

$$A_i, \quad A_i + x_i, \quad A_{i+1} - y_i, \quad A_{i+1}. \tag{5}$$

Thus, we have $2n$ unknowns, so we must find $2n$ conditions.

We call the following *edges*:

$$\widetilde{A}_i = -A_i + A_{i+1} = p_{i,03} \qquad (i = 0, \ldots, n-1). \tag{6}$$

Then from (3) and (4) we get the derivatives

$$\begin{cases} B_i'(0) = 3x_i \\ B_i''(0) = 6(\widetilde{A}_i - 2x_i - y_i) \end{cases} \qquad (i = 0, \ldots, n-1), \tag{7}$$

and

$$\begin{cases} B_i'(1) = 3y_i \\ B_i''(1) = 6(-\widetilde{A}_i + x_i + 2y_i) \end{cases} \qquad (i = 0, \ldots, n-1). \tag{8}$$

3 Conditions

3.1 Stuart Kent and open C^2-continuous curves

Stuart Kent constructs open C^2-continuous curves. C^2-continuity means that at every anchor A_i $(i = 1, \dots, n-1)$ we have $B'_{i-1}(1) = B'_i(0)$ and $B''_{i-1}(1) = B''_i(0)$. By (7) and (8) these come down to

$$\begin{cases} y_{i-1} = x_i, \\ -\widetilde{A}_{i-1} + x_{i-1} + 2y_{i-1} = \widetilde{A}_i - 2x_i - y_i \end{cases} \quad (i = 1, \dots, n-1). \tag{9}$$

So we have $2(n-1)$ conditions. We need another two. Kent imposes "natural boundary conditions": at the start and end of the curve the second derivatives vanish:

$$B''_0(0) = B''_{n-1}(1) = 0, \tag{10}$$

or

$$\begin{cases} \widetilde{A}_0 - 2x_0 - y_0 = 0, \\ -\widetilde{A}_{n-1} + x_{n-1} + 2y_{n-1} = 0. \end{cases} \tag{11}$$

3.2 Dissatisfaction with equal length handles

At each anchor A_i $(i = 1, ..., n-1)$, C^1-continuity forces that $B'_{i-1}(1) = B'_i(0)$, or $y_{i-1} = x_i$, hence the two handles are of equal length. This may result in curves that are visually not so pleasing (matter of taste of course). I feel that the two handles should be allowed to be of different lengths: If the edge $\widetilde{A}_{i-1}$ is longer than the edge $\widetilde{A}_i$ then y_{i-1} should be longer than x_i; thus, a longer edge should receive a longer handle. To this end, we may try to replace the condition with

$$B'_{i-1}(1) = k_i B'_i(0). \tag{12}$$

where k_i is a suitable real number reflecting the relative lengths of the two edges $(i = 1, \dots, n-1)$. The case $k_1 = \dots = k_{n-1} = 1$ gives C^2-continuity. Experiments suggest that choosing

$$k_i = \left(|\widetilde{A}_{i-1}|/|\widetilde{A}_i|\right)^{1/4} \tag{13}$$

gives nice results.

If we do this, then the curve will no longer be C^1-continuous, and consequently not C^2-continuous either. But it will still be G^1-continuous (that is, smooth: at each anchor the two tangents agree). And instead of C^2-continuity it is natural to require G^2-continuity. This means continuity of curvature, which seems more suitable for what we are doing; after all, it is a condition about the visual appearance of the curve while C^2-continuity is a condition on the parametric representations.

If we require (12) then how can we ensure continuous curvature? From the general expression of the curvature of a curve $(x, y) = f(t)$,

$$\kappa(t) = \frac{f''(t) \times f'(t)}{|f'(t)|^3}, \tag{14}$$

we see that we can restore continuity of curvature by replacing the old condition $B''_{i-1}(1) = B''_i(0)$ with the condition

$$B''_{i-1}(1) = k_i^2 B''_i(0). \tag{15}$$

But yet another modification is possible. We introduce new parameters l_i ($i = 0, \dots, n-1$) and set the condition

$$B''_{i-1}(1) + l_{i-1}B'_{i-1}(1) = k_i^2\big(B''_i(0) - l_i B'_i(0)\big). \tag{16}$$

The extra terms vanish in κ, so this does not change the curvatures at the anchor A_{i-1} but it changes the shape of the curve close by.

Experiments suggest that the following scenario is good in an implementation. We define

$$\begin{cases} k_i = \Big(|\widetilde{A}_{i-1}|/|\widetilde{A}_i|\Big)^{K/4} & (i = 1, \dots, n-1) \\ l_i = L & (i = 0, \dots, n-1) \end{cases} \tag{17}$$

and offer the constants K and L as input parameters, with $K = 1$ and $L = 0$ as good default values. Then K determines how strongly different edge lengths are taken into account, and L determines how tightly the curve bends close to the anchors. C^2-continuity is obtained with $K = 0$.

3.3 Closed G^2-continuous curves

From these considerations we derive now a new equation system to replace the system of Stuart Kent.

We consider first the case of closed curves, not touched upon by Kent. This case serves as a good prototype for all other cases since here we need no boundary conditions.

Assume that in the anchor list $A_0, \dots, A_n$ we have $A_0 = A_n$, so that this is a cycle. As in Section 2.3 we denote by $B_i(t)$ the Bézier arc from A_i to A_{i+1} ($i = 0, \dots, n-1$), and we keep the same notations $x_i, y_i, \widetilde{A}_i$ ($i = 0, \dots, n-1$).

We choose by some rule real numbers

$$\begin{cases} k_0, \dots, k_{n-1}, \\ l_0, \dots, l_{n-1}. \end{cases} \tag{18}$$

We do not fix any method for choosing them but one scenario (17) was suggested in Section 3.2. Note that we must now let i in k_i and l_i run $0, \dots, n-1$ and read indices modulo n.

We can now write directly the new equations to replace (9). First the equations for what happens at the anchors $A_1, \dots, A_{n-1}$: We take conditions (12) and (16) and we insert the derivatives (7) and (8) and obtain

$$\begin{cases} y_{i-1} = k_i x_i, \\ -\widetilde{A}_{i-1} + x_{i-1} + 2y_{i-1} + \frac{1}{2} l_{i-1} y_{i-1} = k_i^2 (\widetilde{A}_i - 2x_i - y_i - \frac{1}{2} l_i x_i) \end{cases} \tag{19}$$

for $i = 1, \dots, n-1$. We notice that denoting

$$L_i = 2 + \tfrac{1}{2} l_i \qquad (i = 0, \dots, n-1) \tag{20}$$

the equation is simplified a little:

$$\begin{cases} y_{i-1} = k_i x_i, \\ -\widetilde{A}_{i-1} + x_{i-1} + L_{i-1} y_{i-1} = k_i^2 (\widetilde{A}_i - L_i x_i - y_i) \end{cases} \tag{21}$$

To obtain a closed G^2-continuous composite Bézier curve we must require that the same conditions (21) hold at the anchor $A_0 = A_n$ as well:

$$\begin{cases} y_{n-1} = k_0 x_0, \\ -\widetilde{A}_{n-1} + x_{n-1} + L_{n-1} y_{n-1} = k_0^2 (\widetilde{A}_0 - L_0 x_0 - y_0), \end{cases} \tag{22}$$

where we have written k_0 and l_0 instead of k_n and l_n.

Of course, we can include (22) in (21) if we let $i = 0, \dots, n-1$ and take indices modulo n. Let us write this once more:

$$\begin{cases} y_{i-1} = k_i x_i, \\ -\widetilde{A}_{i-1} + x_{i-1} + L_{i-1} y_{i-1} = k_i^2 (\widetilde{A}_i - L_i x_i - y_i) \end{cases} \tag{23}$$

where $i = 0, \dots, n-1$ and indices are taken modulo n. We have $2n$ unknowns ($x_0, \dots, x_{n-1}$ and $y_0, \dots, y_{n-1}$) and $2n$ equations (23), so no extra boundary conditions are needed. (Later, in the case of open curves, we shall need some boundary conditions in place of (22).)

3.4 Open G^2-continuous curves, case NN

Now we take the case of G^2-continuous open composite Bézier curves and follow the example of Stuart Kent in the case of C^2-continuous curves. This means that we use Kent's natural boundary condition at both ends; see (10).

Since the curve is open, we allow $A_n \neq A_0$.

We choose somehow real numbers

$$\begin{cases} k_1, \dots, k_{n-1}, \\ l_0, \dots, l_{n-1}. \end{cases} \tag{24}$$

As the equations for what happens at the anchors $A_1, \dots, A_{n-1}$ we have the same equations (21):

$$\begin{cases} y_{i-1} = k_i x_i, \\ -\widetilde{A}_{i-1} + x_{i-1} + L_{i-1} y_{i-1} = k_i^2 (\widetilde{A}_i - L_i x_i - y_i) \end{cases} \tag{25}$$

for $i = 1, \dots, n-1$. But instead of closing the curve cyclically (now generally $A_n \neq A_0$) we keep Stuart Kent's "natural boundary conditions" (11):

$$\begin{cases} \widetilde{A}_0 - 2x_0 - y_0 = 0, \\ -\widetilde{A}_{n-1} + x_{n-1} + 2y_{n-1} = 0. \end{cases} \tag{26}$$

We call this case "NN", meaning that we use the natural boundary condition at both ends.

3.5 Open G^2-continuous curves, case HH

Now we look at another choice for boundary conditions to replace the "natural boundary conditions" (10). We call this case "HH", meaning that the boundary conditions we use come from pre-determining the extreme handle at each end.

We need two extra equations for the unknown vectors x_i and y_i. Suppose we have, in addition to the anchors A_i, some two vectors P_1 and Q_2 and we want the first Bézier arc to have control points A_0, P_1, P_2, A_1 and the last Bézier arc to have control points A_{n-1}, Q_1, Q_2, A_n (for some unspecified P_2 and Q_1). So, the handles at each end of the curve are pre-determined. This means, in particular, that if $A_0 \neq P_1$ and $A_n \neq Q_2$ the tangents of the produced curve at each end are pre-determined. We write $X_0 = -A_0 + P_1$ and $Y_{n-1} = -Q_2 + A_n$ and the boundary conditions are

$$\begin{cases} x_0 = X_0, \\ y_{n-1} = Y_{n-1}. \end{cases} \tag{27}$$

These together with the equation system (25) form now the new equation system.

3.6 Open G^2-continuous curves, case NH

We take now case "NH" which means that we use the natural boundary condition at the start and the alternative boundary condition of pre-determining the handle at the end of the curve.

We assume that we are given a vector Q_2 and we want the last Bézier arc to have control points A_{n-1}, Q_1, Q_2, A_n (for some unspecified Q_1). We denote

$Y_{n-1} = -Q_2 + A_n$. The boundary conditions are combined from (26) and (27):

$$\begin{cases} \widetilde{A}_0 - 2x_0 - y_0 = 0, \\ \qquad\qquad y_{n-1} = Y_{n-1}. \end{cases} \tag{28}$$

Our equation system is now these together with the equation system (25).

3.7 Open G^2-continuous curves, case HN

We take now case "HN" which means that we use the alternative boundary condition (pre-determining the handle) at the start and the natural boundary condition at the end of the curve.

We assume that we are given a vector P_1 and we want the first Bézier arc to have control points A_0, P_1, P_2, A_1 (for some unspecified P_2). We denote $X_0 = -A_0 + P_1$. The boundary conditions are combined from (26) and (27):

$$\begin{cases} \qquad\qquad x_0 = X_0, \\ -\widetilde{A}_{n-1} + x_{n-1} + 2y_{n-1} = 0. \end{cases} \tag{29}$$

The equation system is these together with the equation system (25).

4 Equations

4.1 Closed curve

In the closed case the equations are (23) where we let $i = 0, \ldots, n-1$ and take indices modulo n. We solve the y's,

$$y_i = k_{i+1}x_{i+1} \qquad (i = 0, \ldots, n-1) \tag{30}$$

where the indices are read modulo n. Inserting this to the second equation in (23),

$$x_{i-1} + (L_{i-1}k_i + L_i k_i^2)x_i + k_i^2 k_{i+1}x_{i+1} = \widetilde{A}_{i-1} + k_i^2\widetilde{A}_i \qquad (i = 1, \ldots, n) \tag{31}$$

where the indices are read modulo n. To easify notation we write

$$N_i = L_{i-1}k_i + L_i k_i^2 \qquad (i = 0, \ldots, n-1) \tag{32}$$

where the indices are taken modulo n. As a matrix equation (31) is

$$MX = Y \tag{33}$$

where

$$X = \big(x_0,\ x_1,\ \ldots,\ x_{n-1}\big)^T, \tag{34}$$

$$Y = \left(\widetilde{A}_{n-1} + k_0^2 \widetilde{A}_0,\ \widetilde{A}_0 + k_1^2 \widetilde{A}_1,\ \ldots,\ \widetilde{A}_{n-3} + k_{n-2}^2 \widetilde{A}_{n-2},\ \widetilde{A}_{n-2} + k_{n-1}^2 \widetilde{A}_{n-1}\right)^T, \tag{35}$$

and

$$M = \begin{pmatrix} N_0 & k_0^2 k_1 & & & 1 \\ 1 & N_1 & k_1^2 k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & N_{n-2} & k_{n-2}^2 k_{n-1} \\ k_{n-1}^2 k_0 & & & 1 & N_{n-1} \end{pmatrix}. \tag{36}$$

Zero entries in the matrix are not written. Solving this matrix equation gives us $x_0, \ldots, x_{n-1}$, and then $y_0, \ldots, y_{n-1}$ are computed from (30).

4.2 Open curve, case NN

In the open case we have the equations (25) and (26) to solve. We eliminate the y's by first writing

$$\begin{cases} y_i = k_{i+1} x_{i+1} \qquad (i = 0, \ldots, n-2), \\ y_{n-1} = \frac{1}{2}(\widetilde{A}_{n-1} - x_{n-1}), \end{cases} \tag{37}$$

and inserting these to the rest of the equations. We get easily:

$$\begin{cases} 2x_0 + k_1 x_1 = \widetilde{A}_0, \\ x_{i-1} + N_i x_i + k_i^2 k_{i+1} x_{i+1} = \widetilde{A}_{i-1} + k_i^2 \widetilde{A}_i \quad (i = 1, \ldots, n-2), \\ x_{n-2} + \left(N_{n-1} - \frac{1}{2} k_{n-1}^2\right) x_{n-1} = \widetilde{A}_{n-2} + \frac{1}{2} k_{n-1}^2 \widetilde{A}_{n-1}. \end{cases} \tag{38}$$

We write this as a matrix equation

$$MX = Y \tag{39}$$

where

$$X = \left(x_0,\ x_1,\ \ldots,\ x_{n-1}\right)^T, \tag{40}$$

$$Y = \left(\widetilde{A}_0,\ \widetilde{A}_0 + k_1^2 \widetilde{A}_1,\ \widetilde{A}_1 + k_2^2 \widetilde{A}_2,,\ \ldots,\ \widetilde{A}_{n-3} + k_{n-2}^2 \widetilde{A}_{n-2}, \widetilde{A}_{n-2} + \tfrac{1}{2} k_{n-1}^2 \widetilde{A}_{n-1}\right)^T, \tag{41}$$

and

$$M = \begin{pmatrix} 2 & k_1 & & & \\ 1 & N_1 & k_1^2 k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & N_{n-2} & k_{n-2}^2 k_{n-1} \\ & & & 1 & N_{n-1} - \frac{1}{2} k_{n-1}^2 \end{pmatrix}. \tag{42}$$

This matrix equation gives us $x_0, \ldots, x_{n-1}$, and then $y_0, \ldots, y_{n-1}$ are computed from (37).

4.3 Open curve, case HH

With the variation explained in Section 3.5 we have the anchors $A_0, \ldots, A_n$ and two fixed vectors P_1 and Q_2, and we set $X_0 = -A_0 + P_1$ and $Y_{n-1} = -Q_2 + A_n$. We choose real numbers (24) with some rule. We have the equations (25)

$$\begin{cases} y_{i-1} = k_i x_i, \\ -\widetilde{A}_{i-1} + x_{i-1} + L_{i-1} y_{i-1} = k_i^2 (\widetilde{A}_i - L_i x_i - y_i) \end{cases} \tag{43}$$

for $i = 1, \ldots, n-1$, and the boundary conditions

$$\begin{cases} x_0 = X_0, \\ y_{n-1} = Y_{n-1}. \end{cases} \tag{44}$$

We solve the y's in terms of the x's,

$$\begin{cases} y_i = k_{i+1} x_{i+1} \qquad (i = 0, \ldots, n-2), \\ y_{n-1} = Y_{n-1}. \end{cases} \tag{45}$$

and insert them. Then we have the equations

$$\begin{cases} x_0 = X_0, \\ x_{i-1} + N_i x_i + k_i^2 k_{i+1} x_{i+1} = \widetilde{A}_{i-1} + k_i^2 \widetilde{A}_i \quad (i = 1, \ldots, n-2), \\ x_{n-2} + N_{n-1} x_{n-1} = \widetilde{A}_{n-2} + k_{n-1}^2 \widetilde{A}_{n-1} - k_{n-1}^2 Y_{n-1}. \end{cases} \tag{46}$$

As a matrix equation,

$$MX = Y \tag{47}$$

where

$$X = \big(x_0,\ x_1,\ x_2,\ \ldots,\ x_{n-1}\big)^T, \tag{48}$$

$$Y = \big(X_0,\ \widetilde{A}_0 + k_1^2 \widetilde{A}_1,\ \ldots,\ \widetilde{A}_{n-3} + k_{n-2}^2 \widetilde{A}_{n-2},\ \widetilde{A}_{n-2} + k_{n-1}^2 \widetilde{A}_{n-1} - k_{n-1}^2 Y_{n-1}\big)^T, \tag{49}$$

and

$$M = \begin{pmatrix} 1 & & & & \\ 1 & N_1 & k_1^2 k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & N_{n-2} & k_{n-2}^2 k_{n-1} \\ & & & 1 & N_{n-1} \end{pmatrix}. \tag{50}$$

From the matrix equation we get $x_0, \ldots, x_{n-1}$. The remaining unknowns are obtained from (45).

4.4 Open curve, case NH

With the variation in Section 3.6 we have the anchors $A_0, \dots, A_n$ and a fixed vector Q_2, and we set $Y_{n-1} = -Q_2 + A_n$. We choose real numbers (24) with some rule. We have the equations (25)

$$\begin{cases} y_{i-1} = k_i x_i, \\ -\widetilde{A}_{i-1} + x_{i-1} + L_{i-1} y_{i-1} = k_i^2 (\widetilde{A}_i - L_i x_i - y_i) \end{cases} \tag{51}$$

for $i = 1, \dots, n-1$, and the boundary conditions

$$\begin{cases} \widetilde{A}_0 - 2x_0 - y_0 = 0, \\ y_{n-1} = Y_{n-1}. \end{cases} \tag{52}$$

We solve the y's in terms of the x's,

$$\begin{cases} y_i = k_{i+1} x_{i+1} \qquad (i = 0, \dots, n-2), \\ y_{n-1} = Y_{n-1} \end{cases} \tag{53}$$

and insert them. Then we have the equations

$$\begin{cases} 2x_0 + k_1 x_1 = \widetilde{A}_0, \\ x_{i-1} + N_i x_i + k_i^2 k_{i+1} x_{i+1} = \widetilde{A}_{i-1} + k_i^2 \widetilde{A}_i \quad (i = 1, \dots, n-2), \\ x_{n-2} + N_{n-1} x_{n-1} = \widetilde{A}_{n-2} + k_{n-1}^2 \widetilde{A}_{n-1} - k_{n-1}^2 Y_{n-1}. \end{cases} \tag{54}$$

As a matrix equation,

$$MX = Y \tag{55}$$

where

$$X = \big(x_0,\ x_1,\ x_2,\ \dots,\ x_{n-1}\big)^T, \tag{56}$$

$$Y = \big(\widetilde{A}_0,\ \widetilde{A}_0 + k_1^2 \widetilde{A}_1,\ \dots,\ \widetilde{A}_{n-3} + k_{n-2}^2 \widetilde{A}_{n-2},\ \widetilde{A}_{n-2} + k_{n-1}^2 \widetilde{A}_{n-1} - k_{n-1}^2 Y_{n-1}\big)^T, \tag{57}$$

and

$$M = \begin{pmatrix} 2 & k_1 & & & \\ 1 & N_1 & k_1^2 k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & N_{n-2} & k_{n-2}^2 k_{n-1} \\ & & & 1 & N_{n-1} \end{pmatrix}. \tag{58}$$

From the matrix equation we get $x_0, \dots, x_{n-1}$. The remaining unknowns are obtained from (53).

4.5 Open curve, case HN

With the variation in Section 3.7 we have the anchors $A_0, \ldots, A_n$ and a fixed vector P_1, and we set $X_0 = -A_0 + P_1$. We choose real numbers (24) with some rule. We have the equations (25)

$$\begin{cases} y_{i-1} = k_i x_i, \\ -\widetilde{A}_{i-1} + x_{i-1} + L_{i-1} y_{i-1} = k_i^2 (\widetilde{A}_i - L_i x_i - y_i) \end{cases} \tag{59}$$

for $i = 1, \ldots, n-1$, and the boundary conditions

$$\begin{cases} x_0 = X_0, \\ -\widetilde{A}_{n-1} + x_{n-1} + 2y_{n-1} = 0. \end{cases} \tag{60}$$

We solve the y's in terms of the x's,

$$\begin{cases} y_i = k_{i+1} x_{i+1} \qquad (i = 0, \ldots, n-2), \\ y_{n-1} = \frac{1}{2}(\widetilde{A}_{n-1} - x_{n-1}) \end{cases} \tag{61}$$

and insert them. Then we have the equations

$$\begin{cases} x_0 = X_0, \\ x_{i-1} + N_i x_i + k_i^2 k_{i+1} x_{i+1} = \widetilde{A}_{i-1} + k_i^2 \widetilde{A}_i \quad (i = 1, \ldots, n-2), \\ x_{n-2} + (N_{n-1} - \frac{1}{2} k_{n-1}^2) x_{n-1} = \widetilde{A}_{n-2} + \frac{1}{2} k_{n-1}^2 \widetilde{A}_{n-1}. \end{cases} \tag{62}$$

As a matrix equation,

$$MX = Y \tag{63}$$

where

$$X = \big(x_0,\ x_1,\ x_2,\ \ldots,\ x_{n-1}\big)^T, \tag{64}$$

$$Y = \big(X_0,\ \widetilde{A}_0 + k_1^2 \widetilde{A}_1,\ \ldots,\ \widetilde{A}_{n-3} + k_{n-2}^2 \widetilde{A}_{n-2},\ \widetilde{A}_{n-2} + \tfrac{1}{2} k_{n-1}^2 \widetilde{A}_{n-1}\big)^T, \tag{65}$$

and

$$M = \begin{pmatrix} 1 & & & & \\ 1 & N_1 & k_1^2 k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & N_{n-2} & k_{n-2}^2 k_{n-1} \\ & & & 1 & N_{n-1} - \frac{1}{2} k_{n-1}^2 \end{pmatrix}. \tag{66}$$

From the matrix equation we get $x_0, \ldots, x_{n-1}$. The remaining unknowns are obtained from (61).

5 Algorithm

1. Assume we have at least three points $A_0, \dots, A_n$ in the plane, successive points always distinct, but we may have $A_0 = A_n$ (the case of a closed curve).

2. Decide if we are making a closed curve or an open curve and in the latter case which of the cases NN, HH, NH, HN we have at hand. For HH, NH, or HN we need one or two fixed vectors to set up the boundary conditions. For explanations see the opening paragraphs in Sections 3.3–3.7.

3. Denote $\widetilde{A}_i = -A_i + A_{i+1}$ $(i = 0, \dots, n-1)$.

4. Choose real numbers $k_0, \dots, k_{n-1}$ and $l_0, \dots, l_{n-1}$. You can take (17) as a suggestion. (In the case of an open curve, k_0 has no effect.)

5. Denote (cf. (20) and (32))

$$\begin{cases} L_i = 2 + \frac{1}{2} l_i, \\ N_i = L_{i-1} k_i + L_i k_i^2 \end{cases} \qquad (i = 0, \dots, n-1) \tag{67}$$

where indices are taken moduo n.

6. Solve the matrix equation $MX = Y$ where X, Y, M are given by

- (34)–(36) for a closed curve;
- (40)–(42) for an open curve, case NN;
- (48)–(50) for an open curve, case HH;
- (56)–(58) for an open curve, case NH;
- (64)–(66) for an open curve, case HN.

7. The solution X from Step 6 gives $x_0, \dots, x_{n-1}$. Compute $y_0, \dots, y_{n-1}$ from

- (30) for a closed curve (indices modulo n);
- (37) for an open curve, case NN;
- (45) for an open curve, case HH;
- (53) for an open curve, case NH;
- (61) for an open curve, case HN.

8. Form Bézier arcs $B_i(t)$ $(i = 0, \dots, n-1)$ by letting B_i have control points

$$A_i, \quad A_i + x_i, \quad A_{i+1} - y_i, \quad A_{i+1}. \tag{68}$$

9. From $B_0, \dots, B_{n-1}$ form a composite Bézier curve.

References

[1] Stuart Kent: Building Smooth Paths Using Bézier Curves
https://www.stkent.com/2015/07/03/building-smooth-paths-using-bezier-curves.html